Well-posedness and large deviations for 2-D Stochastic Navier-Stokes equations driven by multiplicative Lévy noise

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This talk is based on:

Z., Brzeźniak, X., Peng, J., Zhai, Well-posedness and large deviations for 2-D Stochastic Navier-Stokes equations with jumps

Outline

Motivation: Why we study SPDEs with jump?

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- Application
- Theory

- There are some real world models in financial, physical and biological phenomena, which can not be well represented by a Gaussian noise. For example, in some circumstances, some large moves and unpredictable events can be captured by jump type noise. In recent years, SPDEs driven by Lévy noise have become extremely popular in modeling these phenomena.

- Application
- Theory

- Compared with the case of the Gaussian noise, SPDEs driven by Lévy noise are drastically different because of the appearance of jumps, such as
  - The time regularity,
  - The Burkholder-Davis-Gundy inequality,
  - The Girsanov Theorem,
  - The ergodicity, Irreducibility, Mixing property and other long-time behaviour,
  - Large Deviation Principles.

In general, all the results and/or techniques available for the SPDEs with Gaussian noise are not always suitable for the treatment of SPDEs with Lévy noise and therefore we require new and different techniques.

The time regularity=continuous or càdlàg modification

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- The time regularity of stochastic processes plays an important role in the study of some finer structural properties, such as strong Markov property, measurability and Doob's stopping theorem.
- The existing results on this topic show that the càdlàg property for the equations with Lévy perturbations is much less frequent than the continuity of the trajectories in the Gaussian case.

Example: Consider the linear evolution equation:

$$X(t) = \sum_{n=1}^{\infty} X_n(t)e_n, \quad dX_n(t) = -\gamma_n X_n(t)dt + dL_n(t), \quad X_n(0) = 0,$$

- H =Hilbert space, $\{e_n,\ n \in \mathbb{N}\}$ is an orthonormal and complete basis,
- $\{\gamma_n,\ n \in \mathbb{N}\}$ are positive constants,
- $\{L_n,\ n \in \mathbb{N}\}$ are independent real-valued Lévy processes,
- $L(t) := \sum_{n=1}^{\infty} L_n(t)e_n$.

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We refer to Z. Brzeźniak, B. Goldys, P. Imkeller, S. Peszat, E. Priola, J. Zabczyk(2010), S. Peszat, J. Zabczyk (2013), E. Priola, J. Zabczyk(2011) and references therein for more details.

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  - the Burkholder-Davis-Gundy inequality,
  - the ergodicity, irreducibility, mixing property and other long-time behaviour,
  - Large Deviation Principles.

The Burkholder-Davis-Gundy inequality

The Burkholder-Davis-Gundy inequality is a basic result/tool in the stochastic integral theory.

- For Cylindrical Wiener processes W,

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \Big| \int_0^t G(u(s))dW(s) \Big|^p \Big) \le C_p \mathbb{E}\Big(\int_0^T |G(u(s))|^2_{L^2_{HS}} ds \Big)^{\frac{p}{2}}, \quad p > 0.$$

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- For the compensated Poisson random measure $\widetilde{N}(dz,dt)$ with intensity measure $\nu(dz)dt$,

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\Big|\int_0^t\int_Z G(u(s),z)\widetilde{N}(dz,ds)\Big|^p\Big)\le C_p\mathbb{E}\Big(\int_0^T\int_Z |G(u(s),z)|^2N(dz,ds)\Big)^{\frac{p}{2}},\quad p>1,$$

or

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\Big|\int_0^t\int_Z G(u(s),z)\widetilde{N}(dz,ds)\Big|^p\Big)$$
$$\le C_p\mathbb{E}\Big(\int_0^T\int_Z |G(u(s),z)|^2\nu(dz)ds\Big)^{\frac{p}{2}}+C_p\mathbb{E}\Big(\int_0^T\int_Z |G(u(s),z)|^p\nu(dz)ds\Big),\quad p>1.$$

Girsanov's Theorem

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$$L_T(g) := \frac{1}{2}\int_0^T \|g(t)\|^2 dt.$$

- For the Poisson random measure case the Girsanov theorem is related to some nonlinear transformations.

$$L_T(g) := \int_0^T \int_Z \Big(g(t,z)\log g(t,z) - g(t,z) + 1 \Big) \nu(dz)\, dt.$$

Theory

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We are interested in SPDEs with jump

- Well-posedness
- Wentzell-Freidlin type large deviation principles

Motivation: Well-posedness

In general, there are three approaches to establish Well-posedness for SPDEs with jump.

- Galerkin approximation methods and local monotonicity arguments
- The cut-off methods and the Banach Fixed Point Theorem
- "Big jump" approximating "Small jump"

The first two approaches had been used in proving the existence and the uniqueness of SPDEs with Gaussian noises, The first two approaches had been used in proving the existence and the uniqueness of SPDEs with Gaussian noises,

- Galerkin approximation methods and local monotonicity arguments:
  W. Liu and M. Röckner(2010),
- The cut-off and the Banach Fixed Point Theorem:
  Z. Brzeźniak and A. Millet(2014),
  A. de Bouard and A. Debussche (1999),

Under the classical local-Lipschitz and the one-sided linear growth assumptions on the coefficients, one can prove the existence and the uniqueness of strong solutions in the probabilistic sense for SPDEs with Gaussian noises.

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- Galerkin approximation methods and local monotonicity arguments:
  Z. Brzeźniak, E. Hausenblas and J. Zhu,(2013)
  Z. Brzeźniak, W. Liu and J. Zhu,(2014)
  for some $p \neq 2$,

$$\int_{\mathrm{Z}} \|G(v,z)\|_{\mathrm{H}}^{p} \nu(dz) \leq K(1+\|v\|_{\mathrm{H}}^{p}).$$

- The cut-off and the Banach Fixed Point Theorem:
  H. Bessaih, E. Hausenblas, P.A. Razafimandimby(2015)
  The authors considered the existence and uniqueness of solutions in PDE sense for stochastic hydrodynamical systems with Lévy noise, including 2-D Navier-Stokes equations.

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  They assumed that the function G is globally Lipischitz in the sense that there exists $K > 0$ such that for $p = 1, 2$,

$$\int_{\mathrm{Z}} \|G(v_1, z) - G(v_2, z)\|_{\mathrm{V}}^{2p} \nu(dz) \leq K \|v_1 - v_2\|_{\mathrm{V}}^{2p}, \quad v_1, v_2 \in \mathrm{V},$$

  and

$$\int_{\mathrm{Z}} \|G(v_1, z) - G(v_2, z)\|_{\mathrm{H}}^{2p} \nu(dz) \leq K \|v_1 - v_2\|_{\mathrm{H}}^{2p}, \quad v_1, v_2 \in \mathrm{H}.$$

But using the same idea to the Lévy case, one needs to assume other conditions on the coefficient G of Lévy noise.

Reason:
Because their approaches used the Burkholder-Davis-Gundy inequality with $p \neq 2$ for the compensated Poisson random measure.

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  This method needs some assumptions on the control of the "small jump", see Z. Dong and Y. Xie (2009).

  There exist measurable subsets $U_m, \ m \in \mathbb{N}$ of Z with $U_m \uparrow \mathrm{Z}$ and $\nu(U_m) < \infty$ such that, for some $k > 0$,

$$\sup_{\|v\|_{\mathrm{H}} \leq k} \int_{U_m^c} \|G(v,z)\|_{\mathrm{H}}^2 \nu(dz) \to 0 \text{ as } m \to \infty.$$

- The similar problems rise when one consider the martingale solutions and the advances that have been made so far. See, for instance,
  - Gaussian cases: F. Flandoli and D. Gatarek(1995),
  - Lévy cases: Z. Dong and J. Zhai(2011), E. Motyl(2014).

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- Under the classical Lipschitz and linear growth assumptions, we established the existence and uniqueness of strong solutions in probability sense and PDE sense for the stochatic 2D Navier-Stokes equations with jump.

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- Under the classical Lipschitz and linear growth assumptions, we established the existence and uniqueness of strong solutions in probability sense and PDE sense for the stochatic 2D Navier-Stokes equations with jump.
- Its abstract setting covers many other PDEs, such as the 2D Magneto-Hydrodynamic Equations, the 2D Boussinesq Model for the Bénard Convection, the 2D Magnetic Bérnard Problem, the 3D Leray α-Model for the Navier-Stokes Equations and several Shell Models of turbulence.

Main idea

In fact, we apply the cut-off methods and the Banach Fixed Point Theorem, which is different from H. Bessaih, E. Hausenblas, P.A. Razafimandimby(2015).

Our method strongly depends on a cutting-off function and a trick, we obtain new *a priori* estimates, and succeed to achieve our goals.

Our second aim is to establish the Freidlin-Wentzell's large deviations principle of the strong solutions **(in PDE sense)** for 2-D stochastic Navier-Stokes equations with Lévy noise, obtained in the first part.

Motivation: Wentzell-Freidlin type large deviation principles

There are some results on the Freidlin-Wentzell's large deviations principle of SPDEs with jump so far, especially following the weak convergence approach introduced by A. Budhiraja, J. Chen and P.Dupuis(2013), A. Budhiraja, P.Dupuis and V. Maroulas(2011) for the case of Poisson random measures.

Related results:

- A. de Acosta(1994,2000)
- J. Feng and T. Kurtz(2006)
- M. Röckner and T, Zhang(2007), T. Xu and T. Zhang(2009)
- A. Swięch and J. Zabczyk(2011)

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- J. Zhai, T. Zhang(2015)
- Z. Dong, J. Xiong, J. Zhai and T. Zhang(2017)

Related results on **Strong solutions in Probability sense** for 2-D stochastic Navier-Stokes equations with Lévy noise:

- T. Xu and T. Zhang(2009):
  Additive noises, Large deviations principle
- J. Zhai, T. Zhang(2015):
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Compared with the existing results in the literature, **our main concern is the strong solutions in PDE sense**, hence we need new *a priori* estimates to establish the tightness of the solutions of the perturbed equations. This is non-trivial.

Motivation: Wentzell-Freidlin type large deviation principles

We prove a Girsanov type theorem for Poisson random measure, and apply this result to prove the well-posedness of the control SPDEs. This is a basic step in applying the weak convergence approach to prove the large deviations for SPDEs.

Although these results have been used in the existing literature, but to the best of our knowledge, have no proof, we give the rigorous proofs. The proof is rather involved and it is of an independent interest.

Introduction

Introduction: the two-dimensional Navier-Stokes equation

Introduction

Consider the two-dimensional Navier-Stokes equation

$$\frac{\partial u(t)}{\partial t} - \nu \Delta u(t) + (u(t) \cdot \nabla) u(t) + \nabla p(t,x) = f(t), \qquad (1)$$

with the conditions

$$\begin{cases} (\nabla \cdot u)(t,x) = 0, & \text{for } x \in D, t > 0, \\ u(t,x) = 0, & \text{for } x \in \partial D, t \geq 0, \\ u(0,x) = u_0(x), & \text{for } x \in D, \end{cases} \qquad (2)$$

- D bounded open domain of $\mathbb{R}^2$ with regular boundary ∂D,
- $u(t,x) : [0,T] \times D \to \mathbb{R}^2$ the velocity field at time t and position x,
- $\nu > 0$ the viscosity,
- $p(t,x) : [0,T] \times D \to \mathbb{R}$ denotes the pressure field,
- f is a deterministic external force.

Introduction

Introduce the following standard spaces:

- $V := \left\{ v \in H_0^1(D;\mathbb{R}^2) : \nabla \cdot v = 0, \text{a.e. in } D \right\}$,

$$\text{Norm: } \|v\|_V := \left(\int_D |\nabla v|^2 dx \right)^{\frac{1}{2}} = \|v\|,$$

- H: the closure of V in the L^2-norm

$$|v|_H := \left(\int_D |v|^2 dx \right)^{\frac{1}{2}} = |v|.$$

Introduction

- the Helmhotz-Hodge projection $P_H: \ L^2(D;\mathbb{R}^2) \to H$.
- the Stokes operator A:

$$Au := -P_H \Delta u, \quad \forall u \in H^2(D;\mathbb{R}^2) \cap V.$$

- the nonlinear operator B:

$$B(u,v) := P_H((u \cdot \nabla)v).$$

Set

$$B(u) := B(u,u).$$

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Set

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By applying the operator P_H to each term of (1), we can rewrite it in the following abstract form:

$$du(t) + Au(t)dt + B(u(t))dt = f(t)dt \quad \text{in } L^2([0,T],V'), \quad (3)$$

with $u(0) = u_0 \in H$.

Well-Posedness

Well-posed for 2-D SNSEs driven by multiplicative Levy noise

Well-Posedness

- Z =locally compact Polish space,
  ν =σ-finite measure on Z,
- Leb_∞ =the Lebesgue measure on $[0,\infty)$,
- η =Poisson random measure on $[0,\infty)\times Z$,
  its intensity measure $\mathrm{Leb}_\infty\otimes\nu$,
- the compensated Poisson random measure $\widetilde{\eta}$:

$$\widetilde{\eta}([0,t]\times O)=\eta([0,t]\times O)-t\nu(O),\ \forall O\in\mathcal{B}(Z):\ \nu(O)<\infty.$$

Well-Posedness

Consider

$$du(t) + Au(t)\,dt + B(u(t))\,dt = f(t)\,dt + \int_Z G(u(t-), z)\widetilde{\eta}(dz, dt),$$
$$u_0 \in H. \tag{4}$$

Problem: Well-Posedness?

Well-Posedness

Our main result 1: **Strong solutions in probability sense**

Assumption: $G : H \times \mathrm{Z} \to H$ is a measurable map

- Lipschitz in H

$$\int_{\mathrm{Z}} \|G(v_1, z) - G(v_2, z)\|_{\mathrm{H}}^2 \nu(dz) \le C\|v_1 - v_2\|_{\mathrm{H}}^2, \quad \forall v_1, v_2 \in \mathrm{H}, \tag{5}$$

- Linear growth in H

$$\int_{\mathrm{Z}} \|G(v, z)\|_{\mathrm{H}}^2 \nu(dz) \le C(1 + \|v\|_{\mathrm{H}}^2), \quad \forall v \in \mathrm{H}. \tag{6}$$

- $u_0 \in \mathrm{H}$ and $f \in L^2_{loc}([0, \infty), \mathrm{V}')$.

Well-Posedness

Our main result 1: **Strong solutions in probability sense**

Result: there exists a unique $\mathbb{F}$-progressively measurable process u such that

(1) $u \in D([0,\infty), H) \cap L^2_{loc}([0,\infty), V)$, $\mathbb{P}$-a.s.,

(2) the following equality holds, for all $t \in [0,\infty)$, $\mathbb{P}$-a.s., in V',

$$\begin{aligned} u(t) &= u_0 - \int_0^t Au(s)\,ds - \int_0^t B(u(s))\,ds + \int_0^t f(s)\,ds \\ &\quad + \int_0^t \int_Z G(u(s-), z)\widetilde{\eta}(dz, ds). \end{aligned}$$

Well-Posedness

The existing results

The existing results in the literature need other assumptions on G,

- Big Jump $\Rightarrow$ All Jump: Z. Dong, Y. Xie, (2009)
  There exist measurable subsets U_m, $m \in \mathbb{N}$ of Z with $U_m \uparrow \mathrm{Z}$ and $\nu(U_m) < \infty$ such that, for some $k > 0$,

$$\sup_{\|v\|_{\mathrm{H}} \leq k} \int_{U_m^c} \|G(v,z)\|_{\mathrm{H}}^2 \nu(dz) \to 0 \text{ as } m \to \infty,$$

- Galerken Approximation: Z. Brzeźniak, E. Hausenblas, J. Zhu, (2013); Z. Brzeźniak, W. Liu, J. Zhu, (2014)
  It is assumed that

$$\int_{\mathrm{Z}} \|G(v,z)\|_{\mathrm{H}}^4 \nu(dz) \leq K(1 + \|v\|_{\mathrm{H}}^4).$$

We get rid of these untypical assumptions, and we need different ideas/techeniques.

Well-Posedness

Our main result 2: **Strong solutions in PDE sense**

Assumption $G : V \times \mathrm{Z} \to V$ is a measurable mapping

- Lipschitz in V

$$\int_{\mathrm{Z}} \|G(v_1, z) - G(v_2, z)\|_{\mathrm{V}}^2 \nu(dz) \le C\|v_1 - v_2\|_{\mathrm{V}}^2, \quad v_1, v_2 \in \mathrm{V},$$

- Linear growth in V

$$\int_{\mathrm{Z}} \|G(v, z)\|_{\mathrm{V}}^2 \nu(dz) \le C(1 + \|v\|_{\mathrm{V}}^2), \quad v \in \mathrm{V}.$$

- Linear growth in H

$$\int_{\mathrm{Z}} \|G(v, z)\|_{\mathrm{H}}^2 \nu(dz) \le C(1 + \|v\|_{\mathrm{H}}^2), \quad \forall v \in \mathrm{H}.$$

- $u_0 \in \mathrm{V}$ and $f \in L^2_{loc}([0, \infty), H)$

Well-Posedness

Our main result 2: **Strong solutions in PDE sense**

Result: there exists a unique $\mathbb{F}$-progressively measurable process u such that

(1) $u \in D([0,\infty),\mathrm{V}) \cap L^2_{loc}([0,\infty),\mathcal{D}(\mathrm{A}))$, $\mathbb{P}$-a.s.,

(2) the following equality in V' holds, for all $t \in [0,\infty)$, $\mathbb{P}$-a.s.:

$$\begin{aligned} u(t) &= u_0 - \int_0^t \mathrm{A}u(s)\,ds - \int_0^t B(u(s))\,ds + \int_0^t f(s)\,ds \\ &\quad + \int_0^t \int_{\mathrm{Z}} G(u(s-),z)\widetilde{\eta}(dz,ds). \end{aligned}$$

Well-Posedness

The existing results

- H. Bessaih, E. Hausenblas, P.A. Razafimandimby(2015)

  The authors considered the existence and uniqueness of solutions defined as above for stochastic hydrodynamical systems with Lévy noise, including 2-D Navier-Stokes equations.

  They assumed that the function G is globally Lipischitz in the sense that there exists $K > 0$ such that for $p = 1, 2$,

$$\int_{\mathrm{Z}} \|G(v_1, z) - G(v_2, z)\|_{\mathrm{V}}^{2p} \nu(dz) \leq K\|v_1 - v_2\|_{\mathrm{V}}^{2p}, \quad v_1, v_2 \in \mathrm{V},$$

  and

$$\int_{\mathrm{Z}} \|G(v_1, z) - G(v_2, z)\|_{\mathrm{H}}^{2p} \nu(dz) \leq K\|v_1 - v_2\|_{\mathrm{H}}^{2p}, \quad v_1, v_2 \in \mathrm{H}.$$

LDP

Wentzell-Freidlin type large deviation principles for the strong solutions **(in PDE sense)** for 2-D stochastic Navier-Stokes equations with Lévy noise

LDP

Problem and the main result

LDP

- Z =locally compact Polish space. Fix $T > 0$.
- $$\mathrm{Z}_T = [0,T] \times \mathrm{Z}, \quad Y = Z \times [0,\infty) \text{ and } Y_T = [0,T] \times \mathrm{Z} \times [0,\infty).$$
- $M_T = \mathcal{M}(\mathrm{Z}_T)$: the space of all non-negative measures ϑ on $(\mathrm{Z}_T, \mathcal{B}(\mathrm{Z}_T))$ such that $\vartheta(K) < \infty$ for every compact subset K of Z_T.

We endow the set M_T with the weakest topology, denoted by $\mathcal{T}(M_T)$, such that for every $f \in C_c(\mathrm{Z}_T)$, where by $C_c(\mathrm{Z}_T)$ we denote the space of continuous functions with compact support, the map

$$M_T \ni \vartheta \mapsto \int_{\mathrm{Z}_T} f(z,s)\vartheta(dz,ds) \in \mathbb{R}$$

is continuous.

Analogously we define $\mathbb{M}_T = \mathcal{M}(Y_T)$ and $\mathcal{T}(\mathbb{M}_T)$.

Both $(M_T, \mathcal{T}(M_T))$ and $(\mathbb{M}_T, \mathcal{T}(\mathbb{M}_T))$ are Polish spaces.

LDP

Denote

$$\bar{\Omega} = \mathbb{M}_T,\ \mathcal{G} := \mathcal{T}(\mathbb{M}_T).$$

There exists a unique probability measure $\mathbb{Q}$ on $(\bar{\Omega}, \mathcal{G})$ on which the canonical/identity map

$$\mathrm{N} : \bar{\Omega} \ni \mathrm{m} \mapsto \mathrm{m} \in \mathbb{M}_T$$

is a Poisson random measure (PRM) on $[0, T] \times Z \times [0, \infty)$ with intensity measure $\mathrm{Leb}(dt) \otimes \nu(dz) \otimes \mathrm{Leb}(dr)$, over the probability space $(\bar{\Omega}, \mathcal{G}, \mathbb{Q})$.

We also introduce the following notation:

$$\begin{aligned}
\mathcal{G}_t &= \text{the } \mathbb{Q}\text{-completion of } \sigma\{\mathrm{N}((0,s] \times A) : s \in [0,t],\ A \in \mathcal{B}(Y)\},\quad t \in [0,T] \\
\mathbb{G} &= (\mathcal{G}_t)_{t \in [0,T]}, \\
\mathcal{P} &= \text{the } \mathbb{G}\text{-predictable } \sigma\text{-field on } [0,T] \times \bar{\Omega}, \\
\bar{\mathbb{A}} &= \text{the class of all } (\mathcal{P} \otimes \mathcal{B}(\mathrm{Z}))\text{-measurable}^1 \text{ function } \varphi : \mathrm{Z}_T \times \bar{\Omega} \to [0,\infty)
\end{aligned}$$

[1] To me precise, $(\mathcal{P} \otimes \mathcal{B}(\mathrm{Z})) \setminus \mathcal{B}[0, \infty)$.

LDP

For every function $\varphi \in \bar{\mathbb{A}}$, let us define a counting process N^φ on $[0,T] \times \mathrm{Z}$ by

$$\mathrm{N}^\varphi((0,t] \times A) \quad := \int_{(0,t]\times A\times(0,\infty)} 1_{[r,\infty)}(\varphi(s,x))\mathrm{N}(dsdxdr), \quad t \in [0,T], A \in \mathcal{B}(\mathrm{Z}).$$

Let us observe that

$$\mathrm{N}^\varphi : \bar{\Omega} \to \mathcal{M}(Z_T) = M_T.$$

Analogously, we define a process $\widetilde{\mathrm{N}}^\varphi$, i.e.

$$\widetilde{\mathrm{N}}^\varphi((0,t] \times A) \quad := \quad \int_{(0,t]\times A\times(0,\infty)} 1_{[r,\infty)}(\varphi(s,x))\widetilde{\mathrm{N}}(dsdxdr), t \in [0,T], A \in \mathcal{B}(\mathrm{Z}).$$

LDP

Let us observe that for any Borel function $f : Z_T \to [0, \infty)$,

$$\int_{(0,t]\times Z} f(s,x)\, \widetilde{\mathrm{N}}^{\varphi}(ds,dx) = \int_{(0,t]\times Z\times(0,\infty)} 1_{\{(s,x,r):r\leq\varphi(s,x)\}} f(s,x)\widetilde{\mathrm{N}}(dsdxdr).$$

Let us also notice that that if φ is a constant function a with value $a \in [0, \infty)$, then

$$\begin{aligned} \mathrm{N}^a((0,t]\times A) &= \mathrm{N}\big((0,t)\times A\times(0,a]\big), \quad t\in[0,T], A\in\mathcal{B}(\mathrm{Z}), \\ \widetilde{\mathrm{N}}^a((0,t]\times A) &= \widetilde{\mathrm{N}}\big((0,t)\times A\times(0,a]\big), \quad t\in[0,T], A\in\mathcal{B}(\mathrm{Z}). \end{aligned}$$

LDP

Let us finish this introduction by the following two simple observations.

Proposition 1

In the above framework, for every $a>0$, the map

$$\mathrm{N}^a:\bar{\Omega}\to\mathcal{M}(Z_T)=M_T \tag{7}$$

is a Poisson random measure on $[0,T]\times Z$ with intensity measure $\mathrm{Leb}(dt)\otimes a\nu(dz)$ and $\widetilde{\mathrm{N}}^a$ is equal to the corresponding compensated Poisson random measure.

Proposition 2

In the above framework, suppose that two functions $\varphi,\psi\in\bar{\mathbb{A}}$, a number $T>0$ and a Borel set $A\subset Z$ are such that

$$\varphi(s,z,\omega)=\psi(s,z,\omega)\text{ for }(s,z,\omega)\in[0,T]\times A\times\bar{\Omega}.$$

Then

$$\mathrm{N}^\varphi((0,t]\times B)=\mathrm{N}^\psi((0,t]\times B),\ \text{ for } t\in[0,T],B\in\mathcal{B}(A). \tag{8}$$

LDP

Let us consider the following SPDEs on the given probability space $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$

$$du^{\varepsilon}(t) + \mathrm{A}u^{\varepsilon}(t)\,dt + B(u^{\varepsilon}(t))\,dt = f(t)\,dt + \varepsilon \int_{\mathrm{Z}} G(u^{\varepsilon}(t-), z)\widetilde{\mathrm{N}}^{1/\varepsilon}(dz, dt),$$
$$u^{\varepsilon}(0) = u_0 \in V. \tag{9}$$

- $\mathrm{N}^{1/\varepsilon}(dz, dt)$ is a Poisson random measure on $[0, T] \times Z$ with intensity measure $1/\varepsilon \mathrm{Leb}(dt) \otimes \nu(dz)$

LDP

Let us consider the following SPDEs on the given probability space $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$

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We have proved that there exists a unique solution u^ε to problem (13)

$$u^\varepsilon \in \Upsilon_T^V := D([0, T], \mathrm{V}) \cap L^2([0, T], \mathcal{D}(\mathrm{A})).$$

LDP

Let us consider the following SPDEs on the given probability space $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$

$$du^{\varepsilon}(t) + \mathrm{A}u^{\varepsilon}(t)\,dt + B(u^{\varepsilon}(t))\,dt = f(t)\,dt + \varepsilon \int_{\mathrm{Z}} G(u^{\varepsilon}(t-), z)\widetilde{\mathrm{N}}^{1/\varepsilon}(dz, dt),$$

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We have proved that there exists a unique solution u^{ε} to problem (13)

$$u^{\varepsilon} \in \Upsilon_T^V := D([0, T], \mathrm{V}) \cap L^2([0, T], \mathcal{D}(\mathrm{A})).$$

Our aim is to establish the LDP for the laws of family $\{u^{\varepsilon}\}_{\varepsilon > 0}$ on Υ_T^V.

LDP

In order to introduce our main result, we need the following notation. Denote, for $N > 0$,

$$\begin{aligned} S^N &= \Big\{g : \mathrm{Z}_T \to [0,\infty) : g \text{ is Borel measurable and } L_T(g) \leq N\Big\}, \\ \mathbb{S} &= \cup_{N\geq 1} S^N, \end{aligned}$$

where for a Borel measurable function $g : \mathrm{Z}_T \to [0,\infty)$ we put

$$L_T(g) := \int_0^T \int_{\mathrm{Z}} \Big(g(t,z)\log g(t,z) - g(t,z) + 1\Big)\nu(dz)\,dt. \qquad (10)$$

A function $g \in S^N$ can be identified with a measure $\nu^g \in M_T$, defined by

$$\nu^g(\mathrm{A}) = \int_{\mathrm{A}} g(t,z)\nu(dz)\,dt, \quad \mathrm{A} \in \mathcal{B}(\mathrm{Z}_T).$$

This identification induces a topology on S^N under which S^N is a compact space. Throughout we use this topology on S^N.

LDP

Let us finally define a set

$$\mathcal{H} := \Big\{ h : \mathrm{Z} \to \mathbb{R} : h \text{ is Borel measurable and there exists } \delta > 0 : \int_\Gamma e^{\delta h^2(z)} \nu(dz) < \infty \text{ for all } \Gamma \in \mathcal{B}(\mathrm{Z}) : \nu(\Gamma) < \infty \Big\}.$$

Assumptions LDP

There exist functions $L_i \in \mathcal{H} \cap L^2(\nu)$, for $i = 1, 2, 3$, such that

- Lipschitz in V

$$\|G(u_1, z) - G(u_2, z)\|_{\mathrm{V}} \leq L_1(z)\|u_1 - u_2\|_{\mathrm{V}}, \quad u_1, u_2 \in \mathrm{V}, \; z \in \mathrm{Z},$$

- Linear growth in V

$$\|G(u, z)\|_{\mathrm{V}} \leq L_2(z)(1 + \|u\|_{\mathrm{V}}), \; u \in \mathrm{V}, \; z \in \mathrm{Z},$$

- Linear growth in H

$$\|G(u, z)\|_{\mathrm{H}} \leq L_3(z)(1 + \|u\|_{\mathrm{H}}), \; u \in \mathrm{H}, \; z \in \mathrm{Z}.$$

- $f \in L^2([0, T]; \mathrm{H})$ and $u_0 \in \mathrm{V}$.

LDP

Our main result

The family $\{u^\varepsilon\}_{\varepsilon>0}$ satisfies a LDP on Υ_T^V with the good rate function I defined by[2]

$$I(k) := \inf\left\{L_T(g) : g \in \mathbb{S}, u^g = k\right\}, \quad k \in \Upsilon_T^V, \tag{11}$$

where for $g \in \mathbb{S}$, u^g is the unique solution of the following deterministic PDE

$$\begin{aligned} &\frac{du^g(t)}{dt} + \mathrm{A}u^g(t) + B(u^g(t)) = f(t) + \int_{\mathrm{Z}} G(u^g(t), z)(g(t,z) - 1)\nu(dz), \\ &u^g(0) = u_0. \end{aligned} \tag{12}$$

[2] By convention, $\inf(\emptyset) = \infty$.

LDP

Definition LDP

$(u^\varepsilon - u^0)$ obeys an LDP on Υ_T^V with rate function I, if it holds that

(a) for each closed subset F of Υ_T^V,

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{Q}\left(u^\varepsilon - u^0 \in F\right) \leq -\inf_{x\in F} I(x);$$

(b) for each open subset G of Υ_T^V,

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{Q}\left(u^\varepsilon - u^0 \in G\right) \geq -\inf_{x\in G} I(x).$$

LDP

a Girsanov type theorem for Poisson random measure

LDP

Let us consider the following SPDEs on the given probability space $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$

$$
\begin{aligned}
&du^{\varepsilon}(t) + \mathrm{A}u^{\varepsilon}(t)\,dt + B(u^{\varepsilon}(t))\,dt = f(t)\,dt + \varepsilon \int_{\mathrm{Z}} G(u^{\varepsilon}(t-), z)\widetilde{\mathrm{N}}^{1/\varepsilon}(dz, dt), \\
&u^{\varepsilon}(0) = u_0 \in V.
\end{aligned} \tag{13}
$$

- $\mathrm{N}^{1/\varepsilon}(dz, dt)$ is a Poisson random measure on $[0, T] \times Z$ with intensity measure $1/\varepsilon \mathrm{Leb}(dt) \otimes \nu(dz)$

LDP

Let us consider the following SPDEs on the given probability space $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$

$$du^\varepsilon(t) + \mathrm{A}u^\varepsilon(t)\,dt + B(u^\varepsilon(t))\,dt = f(t)\,dt + \varepsilon \int_{\mathrm{Z}} G(u^\varepsilon(t-), z)\widetilde{\mathrm{N}}^{1/\varepsilon}(dz, dt),$$
$$u^\varepsilon(0) = u_0 \in V. \tag{13}$$

- $\mathrm{N}^{1/\varepsilon}(dz, dt)$ is a Poisson random measure on $[0, T] \times Z$ with intensity measure $1/\varepsilon \mathrm{Leb}(dt) \otimes \nu(dz)$

We have proved that there exists a unique solution u^ε to problem (13)

$$u^\varepsilon \in \Upsilon_T^V := D([0, T], \mathrm{V}) \cap L^2([0, T], \mathcal{D}(\mathrm{A})).$$

Proof of the main result

By the Yamada-Watanabe Theorem, we infer that there exists a family of $\{\mathcal{G}^\varepsilon\}_{\varepsilon>0}$, where

$$\mathcal{G}^\varepsilon : M_T \to \Upsilon_T^V \text{ is a measurable map}$$

such that for every $\varepsilon > 0$, the following condition holds.

(i) if η is a Poisson random measure on Z_T with intensity $\mathrm{Leb}(dt) \otimes \varepsilon^{-1}\nu(dz)$, on a stochastic basis $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1, \mathbb{F}^1)$, $\mathbb{F}^1 = \{\mathcal{F}_t^1, t \in [0, T]\}$, then the process Y^ε defined by

$$Y^\varepsilon := \mathcal{G}^\varepsilon(\varepsilon\eta)$$

is the unique solution of

$$\begin{aligned} dY^\varepsilon(t) &+ \mathrm{A}Y^\varepsilon(t)\,dt + B(Y^\varepsilon(t))\,dt \\ &= f(t)\,dt + \varepsilon\int_{\mathrm{Z}} G(Y^\varepsilon(t-), z)(\eta(dz, dt) - \varepsilon^{-1}\nu(dz)\,dt), \\ Y^\varepsilon(0) &= u_0. \end{aligned} \tag{14}$$

Proof of the main result

Since by Proposition 1, $N^{\varepsilon^{-1}}$ is a Poisson random measure on Z_T with intensity measure $\mathrm{Leb}(dt) \otimes \varepsilon^{-1}\nu(dz)$, we deduce the following result which will be used later on.

Corollary 1

In the above framework, the unique solution of problem (13) on the probability space $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$ is given by the following equality

$$u^{\varepsilon} := \mathcal{G}^{\varepsilon}(\varepsilon N^{\varepsilon^{-1}}). \tag{15}$$

Proof of the main result

For every $g \in \mathbb{S}$, there is a unique solution $u^g \in \Upsilon_T^V$ of equation (12):

$$\frac{du^g(t)}{dt} + \mathrm{A}u^g(t) + B(u^g(t)) = f(t) + \int_{\mathrm{Z}} G(u^g(t), z)(g(t,z) - 1)\nu(dz),$$
$$u^g(0) = u_0.$$

This allows us to define a map

$$\mathcal{G}^0 : \mathbb{S} \ni g \mapsto u^g \in \Upsilon_T^V. \tag{16}$$

Proof of the main result

To finish the proof of the main result, according to A. Budhiraja, J. Chen and P.Dupuis(2013), it is sufficient to verify two claims. The first one is the following.

Claim-LDP-1. For all $N \in \mathbb{N}$, if $g_n,\ g \in S^N$ are such that $g_n \to g$ as $n \to \infty$, then

$$\mathcal{G}^0(g_n) \to \mathcal{G}^0(g) \ \textit{i.e.} \ u^{g_n} \to u^g \quad \text{in} \quad \Upsilon_T^V.$$

Proof of the main result

In order to state the second claim, we need to introduce some additional notation. Let us fix an increasing sequence $\{K_n\}_{n=1,2,\cdots}$ of compact subsets of Z such that

$$\cup_{n=1}^{\infty} K_n = \mathrm{Z}. \tag{17}$$

Let us put

$$\begin{aligned}\bar{\mathbb{A}}_b = \bigcup_{n=1}^{\infty} \Big\{ \varphi \in \bar{\mathbb{A}} : \varphi(t,x,\omega) \in [\frac{1}{n}, n], \text{ if } (t,x,\omega) \in [0,T] \times K_n \times \bar{\Omega} \\ \text{and } \varphi(t,x,\omega) = 1, \text{ if } (t,x,\omega) \in [0,T] \times K_n^c \times \bar{\Omega} \Big\}.\end{aligned} \tag{18}$$

We also denote

$$\begin{aligned}\mathcal{U}^N &:= \{\varphi \in \bar{\mathbb{A}}_b : \varphi(\cdot,\cdot,\omega) \in S^N, \text{ for } \mathbb{Q}\text{-a.a. } \omega \in \bar{\Omega}\}, \\ \mathcal{U} &= \bigcup_{N=1}^{\infty} \mathcal{U}^N.\end{aligned} \tag{19}$$

Claim-LDP-2. For all $N \in \mathbb{N}$, if $\varepsilon_n \to 0$ and φ_{ε_n}, $\varphi \in \mathcal{U}^N$ is such that φ_{ε_n} converges in law to φ, then

$$\mathcal{G}^{\varepsilon_n}\left(\varepsilon_n N^{\varepsilon_n^{-1}\varphi_{\varepsilon_n}}\right) \text{ converges in law to } \mathcal{G}^0(\varphi) \text{ in } \Upsilon_T^V.$$

Proof of the main result

$$\mathcal{G}^{\varepsilon_n}\left(\varepsilon_n N^{\varepsilon_n^{-1}\varphi_{\varepsilon_n}}\right)?$$

$\varphi_{\varepsilon_n} \equiv 1$ By the definition of $\mathcal{G}^{\varepsilon}$, $\mathcal{G}^{\varepsilon}(\varepsilon N^{\varepsilon^{-1}}) := u^{\varepsilon}$ is the unique solution of problem (13) on the probability space $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$,

$$du^{\varepsilon}(t) + \mathrm{A}u^{\varepsilon}(t)\,dt + B(u^{\varepsilon}(t))\,dt = f(t)\,dt + \varepsilon \int_{\mathrm{Z}} G(u^{\varepsilon}(t-), z)\widetilde{\mathrm{N}}^{1/\varepsilon}(dz, dt),$$

$$u^{\varepsilon}(0) = u_0.$$

Proof of the main result

$$\mathcal{G}^{\varepsilon_n}\left(\varepsilon_n N^{\varepsilon_n{}^{-1}\varphi_{\varepsilon_n}}\right)?$$

$\varphi_{\varepsilon_n} \equiv 1$ By the definition of $\mathcal{G}^\varepsilon$, $\mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}}) := u^\varepsilon$ is the unique solution of problem (13) on the probability space $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$,

$$du^\varepsilon(t) + \mathrm{A}u^\varepsilon(t)\,dt + B(u^\varepsilon(t))\,dt = f(t)\,dt + \varepsilon\int_{\mathrm{Z}} G(u^\varepsilon(t-), z)\widetilde{\mathrm{N}}^{1/\varepsilon}(dz, dt),$$

$$u^\varepsilon(0) = u_0.$$

Our aim is to establish the LDP for the laws of family $\{u^\varepsilon\}_{\varepsilon>0}$ on Υ^V_T.

General φ_{ε_n} $\mathcal{G}^{\varepsilon_n}\left(\varepsilon_n N^{\varepsilon_n{}^{-1}\varphi_{\varepsilon_n}}\right)$?

Proof of the main result

- We prove that the process $\mathcal{G}^{\varepsilon}(\varepsilon N^{\varepsilon^{-1}\varphi_{\varepsilon}})$ is the unique solution of the control SPDE. The key is a Girsanov type theorem for Poisson random measure.

$$\begin{aligned} dX_t^{\varepsilon} + \mathrm{A}X_t^{\varepsilon}dt + B(X_t^{\varepsilon})\,dt &= f(t)\,dt + \int_{\mathrm{Z}} G(X_t^{\varepsilon}, z)(\varphi_{\varepsilon}(t,z) - 1)\nu(dz)\,dt \\ &\quad + \varepsilon \int_{\mathrm{Z}} G(X_{t-}^{\varepsilon}, z)\widetilde{N}^{\varepsilon^{-1}\varphi_{\varepsilon}}(dz, dt), \qquad (20) \\ X_0^{\varepsilon} &= u_0, \end{aligned}$$

Proof of the main result

- We prove that the process $\mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon})$ is the unique solution of the control SPDE. The key is a Girsanov type theorem for Poisson random measure.

$$
\begin{aligned}
dX_t^\varepsilon + \mathrm{A}X_t^\varepsilon dt + B(X_t^\varepsilon)\,dt &= f(t)\,dt + \int_{\mathrm{Z}} G(X_t^\varepsilon, z)(\varphi_\varepsilon(t,z) - 1)\nu(dz)\,dt \\
&\quad + \varepsilon \int_{\mathrm{Z}} G(X_{t-}^\varepsilon, z)\widetilde{N}^{\varepsilon^{-1}\varphi_\varepsilon}(dz, dt), \qquad (20) \\
X_0^\varepsilon &= u_0,
\end{aligned}
$$

Then to get Claim-LDP-2, we only need to consider X^ε.

Proof of the main result

a Girsanov type theorem for Poisson random measure

Lemma 1

Assume that $n \in \mathbb{N}$ and that $\varphi_\varepsilon \in \bar{\mathbb{A}}_{b,n}$. Then there exists an $\bar{\mathbb{A}}_{b,n}$-valued sequence $\left(\psi_m\right)_{m\in\mathbb{N}}$ such that the following properties are satisfied.

(R1) For every m there exist $l \in \mathbb{N}$ and $n_1, \cdots, n_l \in \mathbb{N}$, a partition $0 = t_0 < t_1 < \cdots < t_l = T$ and families

$$\xi_{ij}, \quad i = 1, \cdots, l, \ j = 1, \cdots, n_i,$$
$$E_{ij}, \quad i = 1, \cdots, l, \ j = 1, \cdots, n_i,$$

such that ξ_{ij} is $[\frac{1}{n}, n]$-valued, $\mathcal{G}_{t_{i-1}}$-measurable random variables and, for each $i = 1, \cdots, l$, $\left(E_{ij}\right)_{j=1}^{n_i}$ is a measurable partition of the set K_n, such that

$$\psi_m(t,x,\omega) \quad = \quad 1_{\{0\}}(t) + \sum_{i=1}^{l}\sum_{j=1}^{n_i} 1_{(t_{i-1},t_i]}(t)\xi_{ij}(\omega)1_{E_{ij}}(x) + 1_{K_n^c}(x)1_{(0,T]}(t)$$

for all $(t,z,\omega) \in [0,T] \times Z \times \bar{\Omega}$.

(R2) $\lim_{m\to\infty} \int_0^T |\psi_m(t,x,\omega) - \varphi_\varepsilon(t,x,\omega)|\, dt = 0$, for $\nu \otimes \mathbb{Q}$-a.a. $(x,\omega) \in Z \times \bar{\Omega}$.

Proof of the main result

a Girsanov type theorem for Poisson random measure

Lemma 2 Let us put $\psi_\varepsilon = 1/\varphi_\varepsilon$.

(S1) The process $\mathcal{M}_t^\varepsilon(\psi_\varepsilon)$, $t \geq 0$, defined by

$$\begin{aligned}\mathcal{M}_t^\varepsilon(\psi_\varepsilon) = \exp\Big(& \int_{(0,t]\times \mathrm{Z}\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]} \log(\psi_\varepsilon(s,z))\mathrm{N}(ds,dz,dr) \\ & + \int_{(0,t]\times \mathrm{Z}\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]} \Big(-\psi_\varepsilon(s,z)+1\Big)\nu(dz)\,dsdr\Big) \\ = \quad \exp\Big(& \int_{(0,t]\times K_n\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]} \log(\psi_\varepsilon(s,z))\mathrm{N}(dsdzdr) \\ & + \int_{(0,t]\times K_n\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]} \Big(-\psi_\varepsilon(s,z)+1\Big)\nu(dz)\,dsdr\Big), \quad t\in[0,T],\end{aligned}$$

is an $\mathbb{G}$-martingale on $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$,

(S2) the formula

$$\mathbb{P}_T^\varepsilon(A) = \int_A \mathcal{M}_T^\varepsilon(\psi_\varepsilon)\,d\mathbb{Q}, \quad \forall A\in\mathcal{G}$$

defines a probability measure on $(\bar{\Omega}, \mathcal{G})$,

(S3) the measures $\mathbb{Q}$ and $\mathbb{P}_T^\varepsilon$ are equivalent,

(S4) On $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{P}_T^\varepsilon)$, $\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon}$ has the same law as that of $\varepsilon N^{\varepsilon^{-1}}$ on $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$.

Proof of the main result

Theorem 1

For every process $\varphi_\varepsilon \in \bar{\mathbb{A}}_b$ defined on $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$, the process X^ε defined by

$$X^\varepsilon = \mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon}) \tag{21}$$

is the unique solution of (29).

$$\begin{aligned} dX_t^\varepsilon + \mathrm{A}X_t^\varepsilon\,dt + B(X_t^\varepsilon)\,dt &= f(t)\,dt \\ &+ \varepsilon\int_{\mathrm{Z}} G(X_{t-}^\varepsilon, z)\Big(N^{\varepsilon^{-1}\varphi_\varepsilon}(dz, dt) - \varepsilon^{-1}\nu(dz)\,dt\Big), \\ &= f(t)\,dt + \int_{\mathrm{Z}} G(X_t^\varepsilon, z)(\varphi_\varepsilon(t,z) - 1)\nu(dz)\,dt \\ &\quad + \varepsilon\int_{\mathrm{Z}} G(X_{t-}^\varepsilon, z)\widetilde{N}^{\varepsilon^{-1}\varphi_\varepsilon}(dz, dt), \\ X_0^\varepsilon &= u_0, \end{aligned}$$

Proof of the main result

Why we need Theorem 1?

(S4) On $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{P}^{\varepsilon}_{T})$, $\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon}$ has the same law as that of $\varepsilon N^{\varepsilon^{-1}}$ on $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$.

Proof of the main result

Why we need Theorem 1?

(S4) On $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{P}_T^\varepsilon)$, $\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon}$ has the same law as that of $\varepsilon N^{\varepsilon^{-1}}$ on $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$.

By the definition of $X^\varepsilon := \mathcal{G}^\varepsilon$, the process $\mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon})$ is the unique solution of (21) on $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{P}_T^\varepsilon)$, that is

(C1) X^ε is $\mathbb{G}$-progressively measurable process,

(C2) trajectories of X^ε belong to Υ_T^V $\mathbb{P}_T^\varepsilon$-a.s.,

(C3) the following equality holds, in V', for all $t \in [0, T]$, $\mathbb{P}_T^\varepsilon$-a.s.:

$$\begin{aligned} X^\varepsilon(t) = u_0 - \int_0^t \mathrm{A}X^\varepsilon(s)\, ds - \int_0^t B(X^\varepsilon(s))\, ds \\ + \int_0^t f(s)\, ds + \varepsilon \int_0^t \int_{\mathrm{Z}} G(X^\varepsilon(s-), z)(N^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds) - \varepsilon^{-1}\nu(dz)\, ds). \end{aligned} \tag{22}$$

Proof of the main result

Why we need Theorem 1?

(S4) On $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{P}^{\varepsilon}_{T})$, $\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon}$ has the same law as that of $\varepsilon N^{\varepsilon^{-1}}$ on $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$.

By the definition of $X^\varepsilon := \mathcal{G}^\varepsilon$, the process $\mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon})$ is the unique solution of (21) on $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{P}^{\varepsilon}_{T})$, that is

(C1) X^ε is $\mathbb{G}$-progressively measurable process,

(C2) trajectories of X^ε belong to Υ^V_T $\mathbb{P}^{\varepsilon}_{T}$-a.s.,

(C3) the following equality holds, in V', for all $t \in [0, T]$, $\mathbb{P}^{\varepsilon}_{T}$-a.s.:

$$\begin{aligned} X^\varepsilon(t) &= u_0 - \int_0^t \mathrm{A}X^\varepsilon(s)\,ds - \int_0^t B(X^\varepsilon(s))\,ds \\ &+ \int_0^t f(s)\,ds + \varepsilon \int_0^t \int_{\mathrm{Z}} G(X^\varepsilon(s-), z)(N^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds) - \varepsilon^{-1}\nu(dz)\,ds). \end{aligned} \tag{22}$$

Now we will prove that the process X^ε is the unique solution of (22) on $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$, that is

(C1-0) X^ε is $\mathbb{G}$-progressively measurable process,

(C2-0) trajectories of X^ε belong to Υ^V_T $\mathbb{Q}$-a.s.,

(C3-0) the following equality holds, in V', for all $t \in [0, T]$, $\mathbb{Q}$-a.s.:

$$\begin{aligned} X^\varepsilon(t) &= u_0 - \int_0^t \mathrm{A}X^\varepsilon(s)\,ds - \int_0^t B(X^\varepsilon(s))\,ds \\ &+ \int_0^t f(s)\,ds + \varepsilon \int_0^t \int_{\mathrm{Z}} G(X^\varepsilon(s-), z)(N^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds) - \varepsilon^{-1}\nu(dz)\,ds). \end{aligned} \tag{23}$$

Proof of the main result

Why we need Theorem 1?

(S4) On $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{P}_T^{\varepsilon})$, $\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon}$ has the same law as that of $\varepsilon N^{\varepsilon^{-1}}$ on $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$.

By the definition of $X^\varepsilon := \mathcal{G}^\varepsilon$, the process $\mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon})$ is the unique solution of (21) on $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{P}_T^{\varepsilon})$, that is

(C1) X^ε is $\mathbb{G}$-progressively measurable process,

(C2) trajectories of X^ε belong to Υ_T^V $\mathbb{P}_T^{\varepsilon}$-a.s.,

(C3) the following equality holds, in V', for all $t \in [0, T]$, $\mathbb{P}_T^{\varepsilon}$-a.s.:

$$X^\varepsilon(t) = u_0 - \int_0^t \mathrm{A}X^\varepsilon(s)\,ds - \int_0^t B(X^\varepsilon(s))\,ds + \int_0^t f(s)\,ds + \varepsilon \int_0^t \int_{\mathrm{Z}} G(X^\varepsilon(s-), z)(N^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds) - \varepsilon^{-1}\nu(dz)\,ds). \tag{22}$$

Now we will prove that the process X^ε is the unique solution of (22) on $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$, that is

(C1-0) X^ε is $\mathbb{G}$-progressively measurable process,

(C2-0) trajectories of X^ε belong to Υ_T^V $\mathbb{Q}$-a.s.,

(C3-0) the following equality holds, in V', for all $t \in [0, T]$, $\mathbb{Q}$-a.s.:

$$X^\varepsilon(t) = u_0 - \int_0^t \mathrm{A}X^\varepsilon(s)\,ds - \int_0^t B(X^\varepsilon(s))\,ds + \int_0^t f(s)\,ds + \varepsilon \int_0^t \int_{\mathrm{Z}} G(X^\varepsilon(s-), z)(N^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds) - \varepsilon^{-1}\nu(dz)\,ds). \tag{23}$$

Let us note that despite the fact that the two measures $\mathbb{Q}$ and $\mathbb{P}_T^{\varepsilon}$ are equivalent, the equality (22) does not follow from (21) without an additional justification. We provide this justification.

Proof of the main result

The verification of Claim-LDP-1 and Claim-LDP-2 will be given in the following.

Proof of the main result

The verification of Claim-LDP-1

For all $N \in \mathbb{N}$, let $g_n,\ g \in S^N$ be such that $g_n \to g$ in S^N as $n \to \infty$.

Proof of the main result

The verification of Claim-LDP-1

For all $N \in \mathbb{N}$, let $g_n,\ g \in S^N$ be such that $g_n \to g$ in S^N as $n \to \infty$.

Set u^{g_n} be the solution of the following equation with g replaced by g_n.

$$\frac{du^g(t)}{dt} + \mathrm{A}u^g(t) + B(u^g(t)) = f(t) + \int_{\mathrm{Z}} G(u^g(t), z)(g(t,z) - 1)\nu(dz),$$

$$u^g(0) = u_0.$$

Proof of the main result

The verification of Claim-LDP-1

For all $N \in \mathbb{N}$, let $g_n,\ g \in S^N$ be such that $g_n \to g$ in S^N as $n \to \infty$.

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$$u^g(0) = u_0.$$

By the definition of $\mathcal{G}^0$, $\mathcal{G}^0(g_n) = u^{g_n}$ and $\mathcal{G}^0(g) = u^g$.

Proof of the main result

The verification of Claim-LDP-1

For all $N \in \mathbb{N}$, let $g_n,\ g \in S^N$ be such that $g_n \to g$ in S^N as $n \to \infty$.

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$$\frac{du^g(t)}{dt} + \mathrm{A}u^g(t) + B(u^g(t)) = f(t) + \int_{\mathrm{Z}} G(u^g(t), z)(g(t,z) - 1)\nu(dz),$$
$$u^g(0) = u_0.$$

By the definition of $\mathcal{G}^0$, $\mathcal{G}^0(g_n) = u^{g_n}$ and $\mathcal{G}^0(g) = u^g$.

For simplicity, put $u_n = u^{g_n}$ and $u = u^g$.

Proof of the main result

The verification of Claim-LDP-1

For all $N \in \mathbb{N}$, let $g_n,\ g \in S^N$ be such that $g_n \to g$ in S^N as $n \to \infty$.

Set u^{g_n} be the solution of the following equation with g replaced by g_n.

$$\frac{du^g(t)}{dt} + \mathrm{A}u^g(t) + B(u^g(t)) = f(t) + \int_{\mathrm{Z}} G(u^g(t), z)(g(t,z) - 1)\nu(dz),$$
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By the definition of $\mathcal{G}^0$, $\mathcal{G}^0(g_n) = u^{g_n}$ and $\mathcal{G}^0(g) = u^g$.

For simplicity, put $u_n = u^{g_n}$ and $u = u^g$.

To prove Claim-LDP-1, we will prove that

$$u_n \to u \ \text{ in } \Upsilon^V_T.$$

Proof of the main result

The verification of Claim-LDP-1

We can deduce that

- $\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]}\|u_n(t)\|_{\mathrm{V}}^2+\int_0^T\|u_n(t)\|_{\mathcal{D}(\mathrm{A})}^2\,dt\le C_N.$
- Let us fix $\alpha\in(0,1/2)$.

$$\sup_{n\ge1}\|u_n\|^2_{W^{\alpha,2}([0,T],\mathrm{V}')}\le C_N<\infty.$$

Proof of the main result

The verification of Claim-LDP-1

We can deduce that

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- Let us fix $\alpha\in(0,1/2)$.

$$\sup_{n\ge1}\|u_n\|^2_{W^{\alpha,2}([0,T],\mathrm{V}')}\le C_N<\infty.$$

Since the embedding

$$L^2([0,T],\mathcal{D}(\mathrm{A}))\cap W^{\alpha,2}([0,T],\mathrm{V}')\hookrightarrow L^2([0,T],\mathrm{V})$$

is compact, we infer that there exists $\tilde{u}\in L^2([0,T],\mathcal{D}(\mathrm{A}))\cap L^\infty([0,T],\mathrm{V})$ and a sub-sequence (for simplicity, we also denote it by u_n) such that

Proof of the main result

The verification of Claim-LDP-1

We can deduce that

- $\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \|u_n(t)\|_{\mathrm{V}}^2 + \int_0^T \|u_n(t)\|_{\mathcal{D}(\mathrm{A})}^2\, dt \le C_N$.
- Let us fix $\alpha \in (0, 1/2)$.

$$\sup_{n\ge 1} \|u_n\|_{W^{\alpha,2}([0,T],\mathrm{V}')}^2 \le C_N < \infty.$$

Since the embedding

$$L^2([0,T],\mathcal{D}(\mathrm{A})) \cap W^{\alpha,2}([0,T],\mathrm{V}') \hookrightarrow L^2([0,T],\mathrm{V})$$

is compact, we infer that there exists $\tilde{u} \in L^2([0,T],\mathcal{D}(\mathrm{A})) \cap L^\infty([0,T],\mathrm{V})$ and a sub-sequence (for simplicity, we also denote it by u_n) such that

(P1) $u_n \to \tilde{u}$ weakly in $L^2([0,T],\mathcal{D}(\mathrm{A}))$,

(P2) $u_n \to \tilde{u}$ in the weak* topology of $L^\infty([0,T],\mathrm{V})$,

(P3) $u_n \to \tilde{u}$ strongly in $L^2([0,T],\mathrm{V})$.

Proof of the main result

The verification of Claim-LDP-1

Applying (P1)(P2)(P3) and, refer to J. Zhai, T. Zhang(2015) or A. Budhiraja, J. Chen and P.Dupuis(2013),

For every $\varepsilon > 0$, *there exists* $\beta > 0$ *such if* $A \in \mathcal{B}([0, T])$ *satisfies* $\mathrm{Leb}_{[0,T]}(A) \leq \beta$, *then*

$$\sup_{i=1,2,3} \sup_{h \in S^N} \int_A \int_{\mathrm{Z}} L_i(z)|h(s,z) - 1|\nu(dz)\, ds \leq \varepsilon. \tag{24}$$

we can prove

$$\lim_{n\to\infty} \sup_{i=1,2,3} \sup_{k \in S^N} \int_0^T \|u_n(s) - \tilde{u}(s)\|_{\mathrm{V}} \int_{\mathrm{Z}} L_i(z)|k(s,z) - 1|\nu(dz)\, ds = 0. \tag{25}$$

Proof of the main result

The verification of Claim-LDP-1

Applying (P1)(P2)(P3) and, refer to J. Zhai, T. Zhang(2015) or A. Budhiraja, J. Chen and P.Dupuis(2013),

For every $\varepsilon > 0$, there exists $\beta > 0$ such if $A \in \mathcal{B}([0, T])$ satisfies $\mathrm{Leb}_{[0,T]}(A) \leq \beta$, then

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we can prove

$$\lim_{n\to\infty} \sup_{i=1,2,3} \sup_{k \in S^N} \int_0^T \|u_n(s) - \tilde{u}(s)\|_{\mathrm{V}} \int_{\mathrm{Z}} L_i(z)|k(s,z) - 1|\nu(dz)\, ds = 0. \tag{25}$$

By (P1)(P2)(P3) and (25), we can prove that the limit function $\tilde{u}$ is a solution of

$$\frac{du^g(t)}{dt} + \mathrm{A}u^g(t) + B(u^g(t)) = f(t) + \int_{\mathrm{Z}} G(u^g(t), z)(g(t,z) - 1)\nu(dz),$$

$$u^g(0) = u_0.$$

Proof of the main result

The verification of Claim-LDP-1

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we can prove

$$\lim_{n\to\infty} \sup_{i=1,2,3} \sup_{k \in S^N} \int_0^T \|u_n(s) - \tilde{u}(s)\|_{\mathrm{V}} \int_{\mathrm{Z}} L_i(z)|k(s,z) - 1|\nu(dz)\, ds = 0. \tag{25}$$

By (P1)(P2)(P3) and (25), we can prove that the limit function $\tilde{u}$ is a solution of

$$\frac{du^g(t)}{dt} + \mathrm{A}u^g(t) + B(u^g(t)) = f(t) + \int_{\mathrm{Z}} G(u^g(t), z)(g(t,z) - 1)\nu(dz),$$
$$u^g(0) = u_0.$$

By uniqueness of solution, we infer $\tilde{u} = u^g = u$.

Proof of the main result

The verification of Claim-LDP-1

Hence

(P'1) $u_n \to u$ weakly in $L^2([0,T],\mathcal{D}(\mathrm{A}))$,

(P'2) $u_n \to u$ in the weak* topology of $L^\infty([0,T],\mathrm{V})$,

(P'3) $u_n \to u$ strongly in $L^2([0,T],\mathrm{V})$.

Proof of the main result

The verification of Claim-LDP-1

Hence

(P'1) $u_n \to u$ weakly in $L^2([0,T],\mathcal{D}(\mathrm{A}))$,

(P'2) $u_n \to u$ in the weak* topology of $L^\infty([0,T],\mathrm{V})$,

(P'3) $u_n \to u$ strongly in $L^2([0,T],\mathrm{V})$.

Applying (P'1)(P'2)(P'3) and (25), i.e.

$$\lim_{n\to\infty}\sup_{i=1,2,3}\sup_{k\in S^N}\int_0^T \|u_n(s)-u(s)\|_{\mathrm{V}}\int_{\mathrm{Z}} L_i(z)|k(s,z)-1|\nu(dz)\,ds=0,$$

it is not difficulty to prove

$$u_n \to u \ \text{ in } \Upsilon_T^V.$$

The verification of Claim-LDP-1 is thus complete.

Proof of the main result

The verification of Claim-LDP-2

Let us fix an increasing sequence $\{K_n\}_{n=1,2,\cdots}$ of compact subsets of $\mathbb{Z}$ such that

$$\cup_{n=1}^{\infty} K_n = \mathbb{Z}. \tag{26}$$

Let us put

$$\begin{aligned}\bar{\mathbb{A}}_b = \bigcup_{n=1}^{\infty} \Big\{ & \varphi \in \bar{\mathbb{A}} : \varphi(t,x,\omega) \in [\frac{1}{n}, n], \text{ if } (t,x,\omega) \in [0,T] \times K_n \times \bar{\Omega} \\ & \text{and } \varphi(t,x,\omega) = 1, \text{ if } (t,x,\omega) \in [0,T] \times K_n^c \times \bar{\Omega} \Big\}.\end{aligned} \tag{27}$$

We also denote

$$\begin{aligned}\mathcal{U}^N &:= \{\varphi \in \bar{\mathbb{A}}_b : \varphi(\cdot,\cdot,\omega) \in S^N, \text{ for } \mathbb{Q}\text{-a.a. } \omega \in \bar{\Omega}\}, \\ \mathcal{U} &= \bigcup_{N=1}^{\infty} \mathcal{U}^N.\end{aligned} \tag{28}$$

Claim-LDP-2. For all $N \in \mathbb{N}$, if $\varepsilon_n \to 0$ and φ_{ε_n}, $\varphi \in \mathcal{U}^N$ is such that φ_{ε_n} converges in law to φ, then

$$\mathcal{G}^{\varepsilon_n}\left(\varepsilon_n N^{\varepsilon_n^{-1}\varphi_{\varepsilon_n}}\right) \text{ converges in law to } \mathcal{G}^0(\varphi) \text{ in } \Upsilon_T^V.$$

Proof of the main result

The verification of Claim-LDP-2

$$\mathcal{G}^{\varepsilon_n}\left(\varepsilon_n N^{\varepsilon_n^{-1}\varphi_{\varepsilon_n}}\right)?$$

$\varphi_{\varepsilon_n} \equiv 1$ By the definition of $\mathcal{G}^{\varepsilon}$, $\mathcal{G}^{\varepsilon}(\varepsilon N^{\varepsilon^{-1}}) := u^{\varepsilon}$ is the unique solution of problem (13) on the probability space $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$,

$$du^{\varepsilon}(t) + \mathrm{A}u^{\varepsilon}(t)\,dt + B(u^{\varepsilon}(t))\,dt = f(t)\,dt + \varepsilon \int_{\mathrm{Z}} G(u^{\varepsilon}(t-), z)\widetilde{\mathrm{N}}^{1/\varepsilon}(dz, dt),$$
$$u^{\varepsilon}(0) = u_0.$$

Proof of the main result

The verification of Claim-LDP-2

$$\mathcal{G}^{\varepsilon_n}\left(\varepsilon_n N^{\varepsilon_n^{-1}\varphi_{\varepsilon_n}}\right)?$$

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$$u^\varepsilon(0) = u_0.$$

Our aim is to establish the LDP for the laws of family $\{u^\varepsilon\}_{\varepsilon>0}$ on Υ_T^V.

General φ_{ε_n} $\mathcal{G}^{\varepsilon_n}\left(\varepsilon_n N^{\varepsilon_n^{-1}\varphi_{\varepsilon_n}}\right)$?

Proof of the main result

The verification of Claim-LDP-2

- To prove Claim-LDP-2, we first prove that the process $X^\varepsilon := \mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon})$ is the unique solution of the control SPDE (29). The key is a Girsanov type theorem for Poisson random measure.

$$
\begin{aligned}
dX_t^\varepsilon + \mathrm{A}X_t^\varepsilon\,dt + B(X_t^\varepsilon)\,dt &= f(t)\,dt + \int_{\mathrm{Z}} G(X_t^\varepsilon, z)(\varphi_\varepsilon(t,z)-1)\nu(dz)\,dt \\
&\quad + \varepsilon \int_{\mathrm{Z}} G(X_{t-}^\varepsilon, z)\widetilde{N}^{\varepsilon^{-1}\varphi_\varepsilon}(dz,dt), \qquad (29)\\
X_0^\varepsilon &= u_0,
\end{aligned}
$$

Proof of the main result

The verification of Claim-LDP-2

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&\quad + \varepsilon \int_{\mathrm{Z}} G(X_{t-}^\varepsilon, z)\widetilde{N}^{\varepsilon^{-1}\varphi_\varepsilon}(dz,dt), \qquad (29)\\
X_0^\varepsilon &= u_0,
\end{aligned}
$$

Then to get Claim-LDP-2, we only need to consider X^ε.

Proof of the main result

The verification of Claim-LDP-2

- To prove Claim-LDP-2, we first prove that the process $X^\varepsilon := \mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon})$ is the unique solution of the control SPDE (29). The key is a Girsanov type theorem for Poisson random measure.

$$\begin{aligned} dX^\varepsilon_t + \mathrm{A}X^\varepsilon_t\,dt + B(X^\varepsilon_t)\,dt &= f(t)\,dt + \int_{\mathrm{Z}} G(X^\varepsilon_t, z)(\varphi_\varepsilon(t,z)-1)\nu(dz)\,dt \\ &\quad +\varepsilon\int_{\mathrm{Z}} G(X^\varepsilon_{t-}, z)\widetilde{N}^{\varepsilon^{-1}\varphi_\varepsilon}(dz,dt), \qquad (29) \\ X^\varepsilon_0 &= u_0, \end{aligned}$$

  Then to get Claim-LDP-2, we only need to consider X^ε.

- Next we prove some *a priori* estimates to establish the tightness of X^ε.

Proof of the main result

The verification of Claim-LDP-2

- To prove Claim-LDP-2, we first prove that the process $X^\varepsilon := \mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon})$ is the unique solution of the control SPDE (29). The key is a Girsanov type theorem for Poisson random measure.

$$\begin{aligned} dX_t^\varepsilon + \mathrm{A}X_t^\varepsilon dt + B(X_t^\varepsilon)\,dt &= f(t)\,dt + \int_{\mathrm{Z}} G(X_t^\varepsilon, z)(\varphi_\varepsilon(t,z)-1)\nu(dz)\,dt \\ &\quad + \varepsilon \int_{\mathrm{Z}} G(X_{t-}^\varepsilon, z)\widetilde{N}^{\varepsilon^{-1}\varphi_\varepsilon}(dz,dt), \qquad (29) \\ X_0^\varepsilon &= u_0, \end{aligned}$$

  Then to get Claim-LDP-2, we only need to consider X^ε.

- Next we prove some *a priori* estimates to establish the tightness of X^ε.
- By the Skorokhod representation theorem and a similar arguments as proving Claim-LDP-1, we can get Claim-LDP-2.

Proof of the main result

The verification of Claim-LDP-2

a Girsanov type theorem for Poisson random measure

Lemma 1

Assume that $n \in \mathbb{N}$ and that $\varphi_\varepsilon \in \bar{\mathbb{A}}_{b,n}$. Then there exists an $\bar{\mathbb{A}}_{b,n}$-valued sequence $\left(\psi_m\right)_{m\in\mathbb{N}}$ such that the following properties are satisfied.

(R1) For every m there exist $l \in \mathbb{N}$ and $n_1, \cdots, n_l \in \mathbb{N}$, a partition $0 = t_0 < t_1 < \cdots < t_l = T$ and families

$$\begin{aligned} &\xi_{ij}, \quad i = 1, \cdots, l, \; j = 1, \cdots, n_i, \\ &E_{ij}, \quad i = 1, \cdots, l, \; j = 1, \cdots, n_i, \end{aligned}$$

such that ξ_{ij} is $[\frac{1}{n}, n]$-valued, $\mathcal{G}_{t_{i-1}}$-measurable random variables and, for each $i = 1, \cdots, l$, $\left(E_{ij}\right)_{j=1}^{n_i}$ is a measurable partition of the set K_n, such that

$$\psi_m(t,x,\omega) \quad = \quad 1_{\{0\}}(t) + \sum_{i=1}^{l}\sum_{j=1}^{n_i} 1_{(t_{i-1},t_i]}(t)\xi_{ij}(\omega)1_{E_{ij}}(x) + 1_{K_n^c}(x)1_{(0,T]}(t)$$

for all $(t,z,\omega) \in [0,T] \times Z \times \bar{\Omega}$.

(R2) $\lim_{m\to\infty} \int_0^T |\psi_m(t,x,\omega) - \varphi_\varepsilon(t,x,\omega)|\, dt = 0$, for $\nu \otimes \mathbb{Q}$-a.a. $(x,\omega) \in Z \times \bar{\Omega}$.

Proof of the main result

a Girsanov type theorem for Poisson random measure

Lemma 2 Let us put $\psi_\varepsilon = 1/\varphi_\varepsilon$.

(S1) The process $\mathcal{M}_t^\varepsilon(\psi_\varepsilon)$, $t \geq 0$, defined by

$$\begin{aligned}\mathcal{M}_t^\varepsilon(\psi_\varepsilon) = \exp\Big(\int_{(0,t]\times \mathrm{Z}\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]} &\log(\psi_\varepsilon(s,z))\mathrm{N}(ds,dz,dr)\\ &+\int_{(0,t]\times \mathrm{Z}\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}\Big(-\psi_\varepsilon(s,z)+1\Big)\nu(dz)\,dsdr\Big)\\ = \quad \exp\Big(\int_{(0,t]\times K_n\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]} &\log(\psi_\varepsilon(s,z))\mathrm{N}(dsdzdr)\\ &+\int_{(0,t]\times K_n\times[0,\varepsilon^{-1}\varphi_\varepsilon(s,z)]}\Big(-\psi_\varepsilon(s,z)+1\Big)\nu(dz)\,dsdr\Big),\quad t\in[0,T],\end{aligned}$$

is an $\mathbb{G}$-martingale on $(\bar{\Omega},\mathcal{G},\mathbb{G},\mathbb{Q})$,

(S2) the formula

$$\mathbb{P}_T^\varepsilon(A) = \int_A \mathcal{M}_T^\varepsilon(\psi_\varepsilon)\,d\mathbb{Q},\quad \forall A\in\mathcal{G}$$

defines a probability measure on $(\bar{\Omega},\mathcal{G})$,

(S3) the measures $\mathbb{Q}$ and $\mathbb{P}_T^\varepsilon$ are equivalent,

(S4) On $(\bar{\Omega},\mathcal{G},\mathbb{G},\mathbb{P}_T^\varepsilon)$, $\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon}$ has the same law as that of $\varepsilon N^{\varepsilon^{-1}}$ on $(\bar{\Omega},\mathcal{G},\mathbb{G},\mathbb{Q})$.

Proof of the main result

The verification of Claim-LDP-2

Theorem 1
For every process $\varphi_\varepsilon \in \bar{\mathbb{A}}_b$ defined on $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$, the process X^ε defined by

$$X^\varepsilon = \mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon}) \tag{30}$$

is the unique solution of (29).

$$\begin{aligned} dX_t^\varepsilon + \mathrm{A}X_t^\varepsilon\, dt + B(X_t^\varepsilon)\, dt &= f(t)\, dt \\ &+ \varepsilon \int_{\mathrm{Z}} G(X_{t-}^\varepsilon, z)\Big(N^{\varepsilon^{-1}\varphi_\varepsilon}(dz, dt) - \varepsilon^{-1}\nu(dz)\, dt\Big), \\ &= f(t)\, dt + \int_{\mathrm{Z}} G(X_t^\varepsilon, z)(\varphi_\varepsilon(t,z) - 1)\nu(dz)\, dt \\ &\quad + \varepsilon \int_{\mathrm{Z}} G(X_{t-}^\varepsilon, z)\widetilde{N}^{\varepsilon^{-1}\varphi_\varepsilon}(dz, dt), \\ X_0^\varepsilon &= u_0, \end{aligned}$$

Proof of the main result

Why we need Theorem 1?

(S4) On $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{P}^{\varepsilon}_{T})$, $\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon}$ has the same law as that of $\varepsilon N^{\varepsilon^{-1}}$ on $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$.

By assertion (S4) and the definition of $\mathcal{G}^{\varepsilon}$, we infer that the process $X^{\varepsilon} = \mathcal{G}^{\varepsilon}(\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon})$ is the unique solution of (29) on $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{P}^{\varepsilon}_{T})$, that is

(C1) X^{ε} is $\mathbb{G}$-progressively measurable process,

(C2) trajectories of X^{ε} belong to Υ^{V}_{T} $\mathbb{P}^{\varepsilon}_{T}$-a.s.,

(C3) the following equality holds, in V', for all $t \in [0, T]$, $\mathbb{P}^{\varepsilon}_{T}$-a.s.:

$$
\begin{aligned}
X^{\varepsilon}(t) &= u_0 - \int_0^t \mathrm{A}X^{\varepsilon}(s)\,ds - \int_0^t B(X^{\varepsilon}(s))\,ds \\
&\quad + \int_0^t f(s)\,ds + \varepsilon \int_0^t \int_{\mathrm{Z}} G(X^{\varepsilon}(s-), z)(N^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds) - \varepsilon^{-1}\nu(dz)\,ds).
\end{aligned} \tag{31}
$$

Proof of the main result

Why we need Theorem 1?

(S4) On $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{P}^{\varepsilon}_T)$, $\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon}$ has the same law as that of $\varepsilon N^{\varepsilon^{-1}}$ on $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$.

By assertion (S4) and the definition of $\mathcal{G}^\varepsilon$, we infer that the process $X^\varepsilon = \mathcal{G}^\varepsilon(\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon})$ is the unique solution of (29) on $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{P}^{\varepsilon}_T)$, that is

(C1) X^ε is $\mathbb{G}$-progressively measurable process,
(C2) trajectories of X^ε belong to Υ^V_T $\mathbb{P}^{\varepsilon}_T$-a.s.,
(C3) the following equality holds, in V', for all $t \in [0, T]$, $\mathbb{P}^{\varepsilon}_T$-a.s.:

$$X^\varepsilon(t) = u_0 - \int_0^t \mathrm{A}X^\varepsilon(s)\,ds - \int_0^t B(X^\varepsilon(s))\,ds + \int_0^t f(s)\,ds + \varepsilon\int_0^t\int_{\mathrm{Z}} G(X^\varepsilon(s-),z)(N^{\varepsilon^{-1}\varphi_\varepsilon}(dz,ds) - \varepsilon^{-1}\nu(dz)\,ds). \tag{31}$$

Now we will prove that the process X^ε is the unique solution of (29) on $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$, that is

(C1-0) X^ε is $\mathbb{G}$-progressively measurable process,
(C2-0) trajectories of X^ε belong to Υ^V_T $\mathbb{Q}$-a.s.,
(C3-0) the following equality holds, in V', for all $t \in [0, T]$, $\mathbb{Q}$-a.s.:

$$X^\varepsilon(t) = u_0 - \int_0^t \mathrm{A}X^\varepsilon(s)\,ds - \int_0^t B(X^\varepsilon(s))\,ds + \int_0^t f(s)\,ds + \varepsilon\int_0^t\int_{\mathrm{Z}} G(X^\varepsilon(s-),z)(N^{\varepsilon^{-1}\varphi_\varepsilon}(dz,ds) - \varepsilon^{-1}\nu(dz)\,ds). \tag{32}$$

Proof of the main result

Why we need Theorem 1?

(S4) On $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{P}_T^{\varepsilon})$, $\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon}$ has the same law as that of $\varepsilon N^{\varepsilon^{-1}}$ on $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$.

By assertion (S4) and the definition of $\mathcal{G}^{\varepsilon}$, we infer that the process $X^{\varepsilon} = \mathcal{G}^{\varepsilon}(\varepsilon N^{\varepsilon^{-1}\varphi_\varepsilon})$ is the unique solution of (29) on $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{P}_T^{\varepsilon})$, that is

(C1) X^{ε} is $\mathbb{G}$-progressively measurable process,

(C2) trajectories of X^{ε} belong to Υ_T^V $\mathbb{P}_T^{\varepsilon}$-a.s.,

(C3) the following equality holds, in V', for all $t \in [0, T]$, $\mathbb{P}_T^{\varepsilon}$-a.s.:

$$X^{\varepsilon}(t) = u_0 - \int_0^t \mathrm{A}X^{\varepsilon}(s)\,ds - \int_0^t B(X^{\varepsilon}(s))\,ds + \int_0^t f(s)\,ds + \varepsilon\int_0^t \int_{\mathrm{Z}} G(X^{\varepsilon}(s-), z)(N^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds) - \varepsilon^{-1}\nu(dz)\,ds). \tag{31}$$

Now we will prove that the process X^{ε} is the unique solution of (29) on $(\bar{\Omega}, \mathcal{G}, \mathbb{G}, \mathbb{Q})$, that is

(C1-0) X^{ε} is $\mathbb{G}$-progressively measurable process,

(C2-0) trajectories of X^{ε} belong to Υ_T^V $\mathbb{Q}$-a.s.,

(C3-0) the following equality holds, in V', for all $t \in [0, T]$, $\mathbb{Q}$-a.s.:

$$X^{\varepsilon}(t) = u_0 - \int_0^t \mathrm{A}X^{\varepsilon}(s)\,ds - \int_0^t B(X^{\varepsilon}(s))\,ds + \int_0^t f(s)\,ds + \varepsilon\int_0^t \int_{\mathrm{Z}} G(X^{\varepsilon}(s-), z)(N^{\varepsilon^{-1}\varphi_\varepsilon}(dz, ds) - \varepsilon^{-1}\nu(dz)\,ds). \tag{32}$$

Let us note that despite the fact that the two measures $\mathbb{Q}$ and $\mathbb{P}_T^{\varepsilon}$ are equivalent, the equality (32) does not follow from (31) without an additional justification. We provide this justification.

Proof of the main result

The verification of Claim-LDP-2

some *a priori* estimates for X^ε

- for every $\varepsilon \in (0, \varepsilon_N]$ and fix $\alpha \in (0, 1/2)$

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|X^\varepsilon(t)\|^2_{\mathrm{H}} + \int_0^T \|X^\varepsilon(t)\|^2_V \, dt \Big) \leq C_N, \tag{33}$$

$$\mathbb{E}\Big(\|X^\varepsilon\|^2_{W^{\alpha,2}([0,T],\mathrm{V}')} \Big) \leq C_{\alpha,N}. \tag{34}$$

Proof of the main result

The verification of Claim-LDP-2

some *a priori* estimates for X^ε

- for every $\varepsilon \in (0, \varepsilon_N]$ and fix $\alpha \in (0, 1/2)$

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|X^\varepsilon(t)\|^2_{\mathrm{H}} + \int_0^T \|X^\varepsilon(t)\|^2_V \, dt \Big) \leq C_N, \tag{33}$$

$$\mathbb{E}\Big(\|X^\varepsilon\|^2_{W^{\alpha,2}([0,T],\mathrm{V}')} \Big) \leq C_{\alpha,N}. \tag{34}$$

Let us define a stopping time $\tau_{\varepsilon,M}$ by

$$\tau_{\varepsilon,M} := \inf\{t \geq 0 : \sup_{s\in[0,t]} \|X^\varepsilon(s)\|^2_{\mathrm{H}} + \int_0^t \|X^\varepsilon(s)\|^2_{\mathrm{V}} \, ds \geq M\}, \quad M > 0. \tag{35}$$

$$\begin{aligned} \mathbb{Q}(\tau_{\varepsilon,M} < T) &\leq \frac{C_N}{M}, \quad M > 0, \\ \mathbb{Q}(\|X^\varepsilon\|^2_{W^{\alpha,2}([0,T],\mathrm{V}')} \geq R^2) &\leq \frac{C_{\alpha,N}}{R^2}, \quad R, M > 0. \end{aligned} \tag{36}$$

Proof of the main result

The verification of Claim-LDP-2

some *a priori* estimates for X^ε

-
$$\sup_{\varepsilon\in(0,\varepsilon_{N,M})}\mathbb{E}\Big(\sup_{t\in[0,T]}\|X^\varepsilon(t\wedge\tau_{\varepsilon,M})\|_{\mathrm{V}}^2+\int_0^{t\wedge\tau_{\varepsilon,M}}\|X^\varepsilon(s)\|_{\mathcal{D}(\mathrm{A})}^2\,ds\Big)\Big) =: C_{N,M}. \quad (37)$$

Proof of the main result

The verification of Claim-LDP-2

Tightness for X^ε and Y^ε

- The laws of the sequence $\{X^\varepsilon\}$ are tight on the Hilbert space $L^2([0,T],V)$.
- For some $\varrho > 1$, the laws of the sequence $\{X^\varepsilon\}$ are tight on the Skorokhod space $D([0,T],\mathcal{D}(\mathrm{A}^{-\varrho}))$.

Proof of the main result

The verification of Claim-LDP-2

Tightness for X^ε and Y^ε

- The laws of the sequence $\{X^\varepsilon\}$ are tight on the Hilbert space $L^2([0,T],V)$.
- For some $\varrho > 1$, the laws of the sequence $\{X^\varepsilon\}$ are tight on the Skorokhod space $D([0,T],\mathcal{D}(\mathrm{A}^{-\varrho}))$.

For each ε let Y^ε be the unique solution of the following (auxiliary) stochastic Langevin equation:

$$Y^\varepsilon(t) = \int_0^t \mathrm{A} Y^\varepsilon(s)\,ds + \varepsilon \int_0^t \int_{\mathrm{Z}} G(X^\varepsilon(s-),z)\widetilde{N}^{\varepsilon^{-1}\varphi_\varepsilon}(dz,ds).$$

We have

- If $\eta > 0$, then

$$\lim_{\varepsilon\to 0} \mathbb{Q}\Big(\sup_{t\in[0,T]} \|Y^\varepsilon(t)\|_{\mathrm{V}}^2 + \int_0^T \|Y^\varepsilon(s)\|_{\mathcal{D}(\mathrm{A})}^2\,ds \geq \eta \Big) = 0. \qquad (38)$$

Proof of the main result

Conclusion of the proof of Claim-LDP-2

Let φ_{ε_n}, $\varphi \in \mathcal{U}^N$ be such that φ_{ε_n} converges in law to φ as $\varepsilon_n \to 0$. Then the sequence of processes

$$\mathcal{G}^{\epsilon_n}(\varepsilon_n N^{\varepsilon_n{}^{-1}\varphi_{\varepsilon_n}})$$

converges in law on Υ_T^V to a process

$$\mathcal{G}^0(\varphi).$$

Proof of the main result

Conclusion of the proof of Claim-LDP-2

- By Theorem 1,

$$X^{\varepsilon_n} := \mathcal{G}^{\epsilon_n}(\varepsilon_n N^{\varepsilon_n^{-1}\varphi_{\varepsilon_n}})$$

,

- the laws of the processes $\{X^{\varepsilon_n}\}_{n\in\mathbb{N}}$ are tight on $L^2([0,T],\mathrm{V})\cap D([0,T],\mathcal{D}(\mathrm{A}^{-\varrho}))$,
- $\{Y^{\varepsilon_n}\}_{n\in\mathbb{N}}$ converges in probability to 0 in Υ_T^V.

Set

$$\Gamma = \left[L^2([0,T],\mathrm{V})\cap D([0,T],\mathcal{D}(\mathrm{A}^{-\varrho}))\right]\otimes\Upsilon_T^V\otimes S^N.$$

Let $(X,0,\varphi)$ be any limit point of the tight family $\{(X^{\varepsilon_n},Y^{\varepsilon_n},\varphi_{\varepsilon_n}),\ n\in\mathbb{N}\}$.

Proof of the main result

Conclusion of the proof of Claim-LDP-2

By the Skorokhod representation theorem, there exists a stochastic basis $(\Omega^1, \mathbb{F}^1, \mathbb{P}^1)$ and, on this basis, Γ-valued random variables $(X_1, 0, \varphi_1)$, $(X_1^n, Y_1^n, \varphi_1^n)$, $n \in \mathbb{N}$ such that

(a) $(X_1, 0, \varphi_1)$ has the same law as $(X, 0, \varphi)$,

(b) for any $n \in \mathbb{N}$, $(X_1^n, Y_1^n, \varphi_1^n)$ has the same law as $(X^{\varepsilon_n}, Y^{\varepsilon_n}, \varphi_{\varepsilon_n})$,

(c) $\lim_{n\to\infty}(X_1^n, Y_1^n, \varphi_1^n) = (X_1, 0, \varphi_1)$ in Γ, $\mathbb{P}^1$-a.s..

Proof of the main result

Conclusion of the proof of Claim-LDP-2

From the equation satisfied by $(X^{\varepsilon_n}, Y^{\varepsilon_n}, \varphi_{\varepsilon_n})$, we see that $(X_1^n, Y_1^n, \varphi_1^n)$ satisfies

$$\begin{aligned} X_1^n(t) &- Y_1^n(t) \\ &= u_0 - \int_0^t \mathrm{A}(X_1^n(s) - Y_1^n(s))\,ds - \int_0^t B(X_1^n(s))\,ds + \int_0^t f(s)\,ds \\ &+ \int_0^t \int_{\mathrm{Z}} G(X_1^n(s), z)(\varphi_1^n(s,z) - 1)\nu(dz)\,ds,\ \ t \in [0, T]. \end{aligned}$$

$$\lim_{n\to 0}\Big(\sup_{t\in[0,T]} \|Y_1^n(t)\|_{\mathrm{V}}^2 + \int_0^T \|Y_1^n(s)\|_{\mathcal{D}(\mathrm{A})}^2\,ds \Big) = 0, \quad \mathbb{P}^1\text{-a.s.}, \tag{39}$$

Proof of the main result

Conclusion of the proof of Claim-LDP-2

Applying similar arguments in the proof of Claim-LDP-1, we can get Claim-LDP-2.

THANKS!